Technical Appendices and Supplementary Material

Contents

1 Details of subset of MATH training data

In Table 1, we present the original and sampled sizes of the MATH dataset used in our experiments, broken down by subject domain and difficulty level. Sampling was performed uniformly at random within each group to ensure representative coverage of the various topics and levels.

Table 1. **Statistics of the original and sampled MATH dataset by subject domain (left) and by difficulty level (right).** Samples were drawn uniformly at random within each group to ensure representative coverage across topics and difficulty tiers.

By Domain			**By Difficulty Level**		
Domain	#Total	#Sampled	Level	#Total	#Sampled
Algebra	1744	84	Level 1	566	26
Counting and Probability	771	36	Level 2	1348	65
Geometry	870	43	Level 3	1592	77
Intermediate Algebra	1295	63	Level 4	1690	81
Number Theory	869	41	Level 5	2304	111
Prealgebra	1205	58			
Precalculus	746	35			
#Sampled / #Total			***300 / 7500***		

2 Examples of Concept Dependency Graph

Figure 1 presents the concept dependency graphs constructed during the data decomposition process. We observe that an atomic mathematical operation, such as *Addition*, has many edges linking it to more advanced operations.

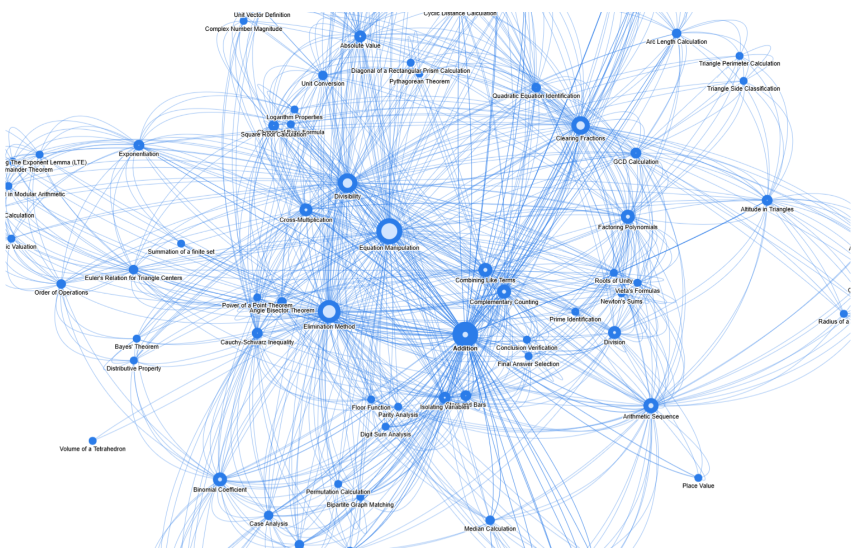

Figure 1. **Concept dependency graph constructed during the AIME data decomposition process.** Nodes represent mathematical concepts, and edges indicate prerequisite relationships between concepts.

3 Zero-shot Small Model Performance

We partitioned the decomposed AIME2024 dataset into five equal-sized bins (quintiles) based on our proposed difficulty measurement, shown in Table 2. This measurement is derived from the concept dependency graph, designed to reflect the conceptual complexity of each problem. We then evaluated the zero-shot performance of the Qwen3-4B-Base model on each difficulty tier.

Our results shown in Figure 2 demonstrate a clear inverse correlation between difficulty score and model accuracy: the model achieves the highest accuracy on problems with the lowest difficulty scores (Quintile 1), and its performance degrades as the difficulty increases, reaching the lowest accuracy on the highest difficulty tier (Quintile 5). This performance trend validates the effectiveness of our concept dependency graph-based difficulty metric in capturing the relative hardness of mathematical problems.

Table 2. **Definition of difficulty quintiles based on concept dependency graph scores.** Each quintile groups problems whose scores fall within the specified range.

Quintile	Difficulty Score Range
Quintile 1 (Easiest)	2.0 – 4.0
Quintile 2	4.0 – 4.0
Quintile 3	4.0 – 6.0
Quintile 4	6.0 – 10.0
Quintile 5 (Hardest)	10.0 – 20.0

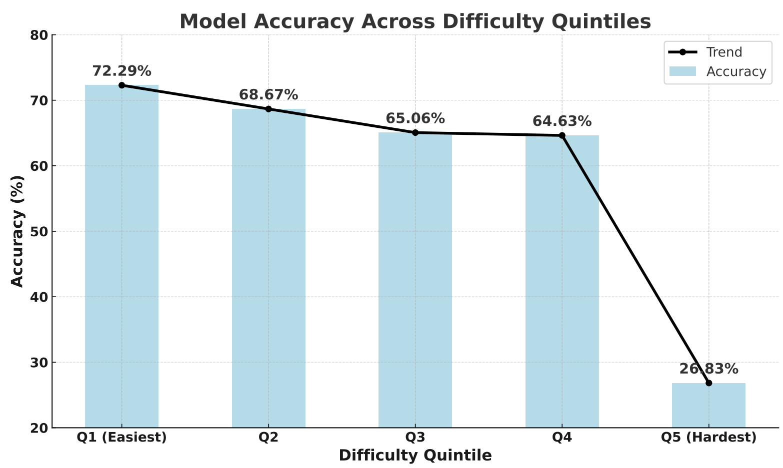

Figure 2. **Zero-shot performance of the Qwen3-4B-Base model across difficulty quintiles.** Accuracy decreases as problem difficulty increases, validating our proposed difficulty metric.

4 Tag Clustering Details

We summarize the tag information identified in both the MATH dataset (Table 3) and the AIME dataset (Tables 4 and 5).

Table 3. **Summary of Cluster Tags**. Sample count represents the number of mathematical problems associated with each cluster tag. Keywords are derived from the tags within each cluster.

Domain	#questions	Keywords
Equation Manipulation	17	Equation Manipulation, Equation Simplification, Equation Solving, Equation Subtraction, Linear Equation, Linear Equation in Two Variables, Linear Equations, Polynomial Equation Solving, Polynomial Simplification, Quadratic Equation, Rational Equation, Rational Equation Solving, Rearranging Equations, Simplifying Expressions, Simultaneous Equations, Simultaneous Equations Solving, Solving Rational Equations, Substitution, Subtraction, Variable substitution
Elimination Method	14	Elimination Method, Equation Solving: Isolating Variables, Linear Equation Solving, Solving Linear Equations, Substitution Method
Addition	12	Addition, Addition of Integers, Arithmetic Addition, Arithmetic Operations, Column Addition, Digit Sum, Fraction Addition, Integer Addition, Multiplication, Place Value Addition, Summation
Divisibility	11	Divisibility, Divisibility Rules, Division Property of Equality, Polynomial Division
Clearing Fractions	10	Clearing Fractions, Common Denominator Calculation, Fraction Multiplication, Fraction Simplification, Fraction Subtraction, Partial Fraction Decomposition, Reciprocal Calculation, Simplifying Fractions, Subtracting Fractions with Common Denominator
Arithmetic Sequence	7	Arithmetic Sequence, Arithmetic Sequence Formula, Arithmetic Subtraction, Counting Integers in an Arithmetic Sequence, Modular Arithmetic, Sum of a Sequence, Sum of an Arithmetic Series
Factoring Polynomials	6	Factoring Polynomials, Factoring Quadratic Equations, Factoring by Grouping, Factoring by grouping, Polynomial Expansion, Prime Factorization
Binomial Coefficient	6	Binomial Coefficient, Binomial Coefficient Calculation, Combination Formula, Combinations Calculation, Combinatorial Counting, Sum of coefficients
Complementary Counting	6	Complementary Counting, Counting Principle, Counting Principles, Counting Rows, Inclusion-Exclusion Principle
Combining Like Terms	6	Combining Like Terms
Cross-Multiplication	5	Cross-Multiplication, Multiplication Principle, Prime Multiplication, Scalar Multiplication
Division	5	Division, Division of Equations, Division of constants, Long Division
Absolute Value	4	Absolute Value, Absolute Value Calculation, Absolute Value Equation, Absolute Value Equation Solving, Absolute Value Equations, Magnitude of a Complex Number, Magnitude of a Vector
Isolating Variables	4	Isolating Variables, Isolating the Variable
GCD Calculation	3	GCD Calculation, GCD Property
Cauchy-Schwarz Inequality	3	Cauchy-Schwarz Inequality, Compound Inequalities, Inequalities, Inequality Manipulation, Linear Inequality Simplification
Altitude in Triangles	3	Altitude in Triangles, Altitude of a Triangle, Similar Triangles, Triangle Construction
Stars and Bars	3	Stars and Bars, Stars and Bars Method
Exponentiation	3	Exponentiation, Logarithm Power Rule, Modular Exponentiation
Square Root Calculation	3	Square Root Calculation, Squaring a number, Squaring both sides
Order of Operations	2	Order of Operations, Order of an Element
Identifying Parallel Lines	2	Identifying Parallel Lines, Line Intersection, Parallel Lines in Polygons, Slope of Parallel and Perpendicular Lines, Slope of Perpendicular Lines
Perpendicular Slopes	2	Perpendicular Slopes, Slope of a Line, Vertical Tangent Line
Quadratic Equation Identification	2	Quadratic Equation Identification, Quadratic Formula
Combinatorial Placement	2	Combinatorial Placement
Solving Linear Inequalities	2	Solving Linear Inequalities, System of Linear Equations
Cyclic Distance Calculation	2	Cyclic Distance Calculation, Distance Formula
Euler’s Relation for Triangle Centers	2	Euler’s Relation for Triangle Centers, Euler’s Theorem
Arc Length Calculation	2	Arc Length Calculation, Perimeter Calculation
Angle Bisector Theorem	2	Angle Bisector Theorem, Perpendicular Bisector of a Chord
Bayes’ Theorem	1	Bayes’ Theorem, Conditional Probability
Distributive Property	1	Distributive Property
Floor Function	1	Floor Function
Finding Zeros of a Function	1	Finding Zeros of a Function
Factorial Calculation	1	Factorial Calculation
Pigeonhole Principle	1	Pigeonhole Principle
Conclusion Verification	1	Conclusion Verification
Change of Base Formula	1	Change of Base Formula
Logarithm Properties	1	Logarithm Properties, Logarithmic Identity, Logarithmic Properties, Logarithmic Reciprocity
Summation of a finite set	1	Summation of a finite set
Complex Number Magnitude	1	Complex Number Magnitude, Complex Number Scaling, Dot Product Magnitude, Modulus of a Complex Number, Vector Magnitude Calculation
Place Value	1	Place Value
Chinese Remainder Theorem	1	Chinese Remainder Theorem
Lifting The Exponent Lemma (LTE)	1	Lifting The Exponent Lemma (LTE)
Order of an Element in Modular Arithmetic	1	Order of an Element in Modular Arithmetic, Order of an element modulo p
Primitive Root Calculation	1	Primitive Root Calculation
p-adic Valuation	1	p-adic Valuation, v_p (p-adic valuation)
Diagonal of a Rectangular Prism Calculation	1	Diagonal of a Rectangular Prism Calculation
Pythagorean Theorem	1	Pythagorean Theorem, Pythagorean Theorem in 3D
Interior Angle of a Regular Polygon	1	Interior Angle of a Regular Polygon
Pairing Elements	1	Pairing Elements
Power of a Point Theorem	1	Power of a Point Theorem
Bipartite Graph Matching	1	Bipartite Graph Matching
Permutation Calculation	1	Permutation Calculation
Prime Identification	1	Prime Identification, Prime Number Identification, Prime Number Multiplication
Triangle Perimeter Calculation	1	Triangle Perimeter Calculation
Triangle Side Classification	1	Triangle Side Classification
Counterexample	1	Counterexample
Mode Calculation	1	Mode Calculation, Mode Identification
Sorting Numbers	1	Sorting Numbers
Area of a Circle	1	Area of a Circle, Area of a Circle Calculation
Cross-Section of a Sphere	1	Cross-Section of a Sphere, Cross-sections of spheres
LCM Calculation	1	LCM Calculation
Proportion	1	Proportion, Proportion Solving, Ratio and Proportion
Radius of a sphere	1	Radius of a sphere
Volume of a Tetrahedron	1	Volume of a Tetrahedron
Newton’s Sums	1	Newton’s Sums
Vieta’s Formulas	1	Vieta’s Formulas, Vieta’s formulas

Table 4. **Summary of Cluster Tags (Sample Count ≥ 3)**. Sample count represents the number of mathematical problems associated with each cluster tag. Keywords are derived from the tags within each cluster.

Domain	#questions	Keywords
Combinatorial Probability	100	Combinatorial Probability, Counting & Probability
Basic Counting Principle	46	Basic Counting Principle, Counting Principle, Counting Principles, Fundamental Counting Principle, Fundamental Principle of Counting, Multiplication Principle of Counting
Combination Calculation	36	Combination Calculation, Combination Enumeration, Combination Formula, Combination Selection, Combination Subtraction, Combination and Permutation Calculation, Combinations, Combinations Calculation, Combinations Formula, Counting Combinations
Factorial	31	Factorial, Factorial Calculation, Factorial Division, Factorial Expansion, Factorial Manipulation, Factorial Multiplication, Factorial Properties, Factorial Simplification, Factorial simplification
Combinatorial Counting	30	Combinatorial Counting, Combinatorial Enumeration, Combinatorial Exclusion, Combinatorial Reasoning, Combinatorics, Counting, Counting Arrangements, Counting Multiples, Counting Subsets
Binomial Coefficient	29	Binomial Coefficient, Binomial Coefficient Calculation, Binomial Coefficient Formula, Binomial Coefficient Multiplication, Binomial Coefficient Simplification, Binomial Coefficients, Binomial Coefficients Calculation, Binomial Expansion, Binomial Probability Formula, Binomial Theorem
Fraction Conversion	28	Fraction Conversion, Fraction Identification, Fraction Multiplication, Fraction Subtraction, Multiplication of Fractions, Percentage to Fraction Conversion, Simplifying Fractions
Linear Equation Evaluation	26	Linear Equation Evaluation, Linear Equation Solving, Linear Expression Evaluation, Solving Linear Equations
Division Simplification	24	Division Simplification, Division of Fractions, Fraction Division, Fraction Multiplication and Simplification, Fraction Simplification, Ratio Simplification, Simplifying Ratios
Addition	23	Addition, Addition of Integers, Addition of integers, Arithmetic Addition, Integer Addition
Arithmetic Operations	21	Arithmetic Operations, Basic Arithmetic Subtraction, Multiplication, Multiplication of Integers, Multiplication of integers
Division Property of Equality	21	Division Property of Equality
Exponent Simplification	21	Exponent Simplification, Exponentiation
Basic Probability Calculation	19	Basic Probability Calculation, Conditional Probability Calculation, Probability, Probability Calculation
Equation Substitution	18	Equation Substitution, Substitution, Substitution Method, Variable Substitution
Circular Permutations	17	Circular Permutations, Cyclic Permutations, Permutation, Permutations
Set Subtraction	13	Set Subtraction, Subtraction
Division	12	Division, Division Algorithm, Long Division
GCD Calculation	12	GCD Calculation
Independent Probability Multiplication	11	Independent Probability Multiplication, Multiplication Rule for Independent Events, Multiplication Rule for Probabilities, Probability Multiplication Rule
Isolating Variables	10	Isolating Variables, Isolating the variable
Combinations with Repetition	9	Combinations with Repetition, Permutations and Combinations, Permutations with Repetition
Counting Exclusion	9	Counting Exclusion, Exclusion Principle, Inclusion-Exclusion Principle
Counting and Summation	8	Counting and Summation, Summation
Adding Fractions	8	Adding Fractions, Adding Fractions with Like Denominators, Adding Fractions with Unlike Denominators, Addition of Fractions, Arithmetic with Fractions, Common Denominator Addition, Fraction Addition, Multiplying Fractions
Prime Factorization	8	Prime Factorization
Combining Like Terms	7	Combining Like Terms
Place Value	7	Place Value, Place Value Identification
Arithmetic Sequence	6	Arithmetic Sequence, Arithmetic Sequence Formula, Arithmetic Sequence Identification, Arithmetic Sequence Sum, Arithmetic Sequence Sum Formula, Arithmetic Sequence Summation, Arithmetic Sequences, Arithmetic Sequences Counting, Counting Terms in an Arithmetic Sequence
Equation Simplification	6	Equation Simplification, Linear Equation Simplification, Polynomial Simplification
Conditional Probability	6	Conditional Probability
Permutation with Restrictions	6	Permutation with Restrictions, Permutations with Restrictions, Permutations with restrictions
Divisibility Rule for 3	6	Divisibility Rule for 3, Divisibility Rules
Factoring by grouping	6	Factoring by grouping, Factorization
Complement Rule	5	Complement Rule, Complement Rule in Probability
Multiplication Principle	4	Multiplication Principle
Arithmetic Series Formula	4	Arithmetic Series Formula, Arithmetic Series Sum Formula, Arithmetic Series Summation, Arithmetic Sum Calculation, Summation of Arithmetic Series
Order of Operations	4	Order of Operations
Equation Balancing	3	Equation Balancing
Binomial Probability	3	Binomial Probability
Counting Integers	3	Counting Integers, Counting Integers in a Range
Power Set Calculation	3	Power Set Calculation
Discriminant Calculation	3	Discriminant Calculation
Identifying Coefficients in a Quadratic Equation	3	Identifying Coefficients in a Quadratic Equation, Quadratic Coefficients Identification
Complementary Counting	3	Complementary Counting
Distributive Property	3	Distributive Property
Combination Symmetry	3	Combination Symmetry, Combinatorial Symmetry, Symmetry Counting
Area Calculation	3	Area Calculation, Area Calculation of a Square, Area Ratios, Area of a Rectangle Calculation, Area of a Square Calculation, Area of a Triangle Calculation
Area of a Right Triangle	3	Area of a Right Triangle, Area of a Triangle, Triangle Area Formula
Counting Outcomes	3	Counting Outcomes, Enumerating Outcomes
Independent Events	3	Independent Events, Independent Events Probability, Independent Events Probability Calculation, Probability of Independent Events
Scalar Multiplication	3	Scalar Multiplication
Case Analysis	3	Case Analysis
Common Denominator Calculation	3	Common Denominator Calculation, Common Denominator Conversion, Finding a Common Denominator, Subtracting Fractions with Common Denominator, Subtracting Fractions with Common Denominators
Finding the Least Common Multiple (LCM)	3	Finding the Least Common Multiple (LCM), LCM Calculation, Least Common Multiple (LCM) Calculation
Factoring Common Factor	3	Factoring Common Factor, Factoring Out Common Factors, Finding Factors
Counting Even Numbers	3	Counting Even Numbers, Counting Odd Numbers, Even Numbers Identification, Even and Odd Numbers, Identifying Even Numbers
Modular Arithmetic	3	Modular Arithmetic, Modulo Operation

Table 5. **Summary of Cluster Tags (Sample Count < 3)**. Sample count represents the number of mathematical problems associated with each cluster tag. Keywords are derived from the tags within each cluster.

Domain	#questions	Keywords
Probability Distribution	2	Probability Distribution
Range of Sums for Dice Rolls	2	Range of Sums for Dice Rolls, Sum of Two Dice Rolls
Uniform Probability Distribution	2	Uniform Probability Distribution
Properties of Platonic Solids	2	Properties of Platonic Solids, Symmetry of Platonic Solids
Division of Constants	2	Division of Constants, Division of constants
Permutations of Multisets	2	Permutations of Multisets
Digit Fixation in Positional Notation	2	Digit Fixation in Positional Notation, Digit Placement
Multiples Identification	2	Multiples Identification
Distance Formula	2	Distance Formula, Horizontal Distance Calculation
Graphing Inequalities	2	Graphing Inequalities, Graphing Linear Inequalities, Linear Inequality Graphing
Intersection of Lines and Curves	2	Intersection of Lines and Curves, Line Intersection
Isosceles Right Triangle	2	Isosceles Right Triangle, Isosceles Triangle Properties
Probability of Combined Events	2	Probability of Combined Events, Probability of a Single Event
Combinatorial Selection	2	Combinatorial Selection, Subset Selection
Subset Identification	2	Subset Identification
Complementary Probability	2	Complementary Probability
Probability Addition Rule	2	Probability Addition Rule, Total Probability Rule
Factor Pairing	2	Factor Pairing, Factor Pairs Identification
Finding Multiples	2	Finding Multiples, Multiples of a Number
Prime Identification	2	Prime Identification, Prime Number Identification, Prime and Composite Numbers Identification
Expected Value Calculation	2	Expected Value Calculation
Addition and Subtraction Properties of Equality	2	Addition and Subtraction Properties of Equality
Long Multiplication	2	Long Multiplication, Multiplication of Large Numbers
Geometric Series	2	Geometric Series, Geometric Series Formula, Geometric Series Identification, Geometric Series Sum Formula, Geometric Series Summation, Infinite Geometric Series Formula, Sum of Infinite Geometric Series
Pascal's Triangle Construction	2	Pascal's Triangle Construction, Pascal's Triangle Row Sum
Independent Probability	2	Independent Probability
Exponentiation of Fractions	2	Exponentiation of Fractions
Simplifying Rational Expressions	2	Simplifying Rational Expressions
Pascal's Identity	2	Pascal's Identity, Pascal's Triangle
Summation of Series	2	Summation of Series, Summation of a Sequence
Intersection of Sets	2	Intersection of Sets, Set Intersection
Set Union	2	Set Union, Set Union Cardinality
Percentage Calculation	2	Percentage Calculation, Percentage Conversion, Percentage to Decimal Conversion
Symmetry in Probability	2	Symmetry in Probability
Counting Grid Positions	2	Counting Grid Positions, Counting Rectangles in a Grid, Counting Squares in a Grid, Counting Subsets in a Grid
Parity	2	Parity
Recurrence Relation	2	Recurrence Relation, Recurrence Relations
Block Permutation	2	Block Permutation
Digit Pairing for Sum	2	Digit Pairing for Sum, Digit Sum Calculation, Pairing Numbers for a Fixed Sum
Permutation Calculation	2	Permutation Calculation
Cyclic Number Patterns	2	Cyclic Number Patterns, Cyclic Sequences
Bipartite Graph	2	Bipartite Graph, Bipartite Graph Coloring
Digit Constraints	2	Digit Constraints, Digit Restriction, Digit Sum Constraints, Single-digit constraint
Counting Leap Years	1	Counting Leap Years, Leap Year Calculation
Minimum Value Calculation	1	Minimum Value Calculation
Pigeonhole Principle	1	Pigeonhole Principle
Range Calculation	1	Range Calculation
Burnside's Lemma	1	Burnside's Lemma
Polyhedron Properties	1	Polyhedron Properties, Properties of Polyhedra
Rotational Symmetry	1	Rotational Symmetry
Rotational Symmetry of Polyhedra	1	Rotational Symmetry of Polyhedra, Symmetry in Polyhedra
Slope Calculation	1	Slope Calculation
Pair Counting	1	Pair Counting
Pairwise Sum Calculation	1	Pairwise Sum Calculation
Division of Even Numbers	1	Division of Even Numbers
Frequency Distribution	1	Frequency Distribution
Conditional Statements	1	Conditional Statements
Counting Intervals	1	Counting Intervals
Period Calculation	1	Period Calculation
Time Interval Calculation	1	Time Interval Calculation
Unit Conversion	1	Unit Conversion
Factoring Quadratic Expressions	1	Factoring Quadratic Expressions
Simultaneous Equations	1	Simultaneous Equations
Counting Cyclic Quadrilaterals with Integer Sides	1	Counting Cyclic Quadrilaterals with Integer Sides
Counting Rectangles	1	Counting Rectangles
Hockey Stick Identity	1	Hockey Stick Identity
Perimeter Calculation	1	Perimeter Calculation
Properties of Quadrilaterals	1	Properties of Quadrilaterals, Symmetry in Quadrilaterals
Properties of a Square	1	Properties of a Square
Stars and Bars Method	1	Stars and Bars Method
Triangle Inequality	1	Triangle Inequality
Distance Comparison	1	Distance Comparison
Absolute Value Simplification	1	Absolute Value Simplification
Subset Definition	1	Subset Definition
Equivalent Fractions	1	Equivalent Fractions
Cube Root Estimation	1	Cube Root Estimation
Inequality Comparison	1	Inequality Comparison
Sequential Multiplication	1	Sequential Multiplication
Angle Measurement in Radians	1	Angle Measurement in Radians, Radian-Degree Conversion
Arc Length Calculation	1	Arc Length Calculation, Arc Length Formula
Arc Measure	1	Arc Measure
Central Angle Theorem	1	Central Angle Theorem
Circumference of a Circle	1	Circumference of a Circle
Commutative Property of Addition	1	Commutative Property of Addition
Factorial Decomposition	1	Factorial Decomposition
Matrix Indexing	1	Matrix Indexing
Probability with Replacement	1	Probability with Replacement

5 Examples of Decomposed Data

We apply our data decomposition method to the S1 dataset and demonstrate the two-layer recursive decomposition process for a sample data point in Table 6.

Table 6. **Decomposed S1 data sample.** Black text denotes the original data; yellow text represents the first level of decomposition; and blue text represents the second level of decomposition.

{Question:"Given a rational number, write it as a fraction in lowest terms and calculate the product of the resulting numerator and denominator. For how many rational numbers between 0 and 1 will 20! be the resulting product?",
Solution:"Let the rational number between 0 and 1 be r. When written as a fraction in lowest terms, let it be $\frac{a}{b}$, where a and b are positive integers, $\gcd(a, b) = 1$, and $0 < a < b$.We are given that the product of the numerator and denominator is 20!, so $a \times b = 20!$.We need to find the number of pairs (a, b) such that $a \times b = 20!$, $\gcd(a, b) = 1$, and $a < b$.Let $N = 20!$. The prime factorization of N is $p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, where p_i are distinct prime numbers.The number of distinct prime factors of 20! is the number of primes less than or equal to 20. These primes are 2, 3, 5, 7, 11, 13, 17, 19. Thus, there are 8 distinct prime factors, so $k = 8$.Since $a \times b = N$ and $\gcd(a, b) = 1$, each prime power factor $p_i^{e_i}$ of N must be entirely a factor of a or entirely a factor of b.The number of ways to distribute the prime power factors between a and b is 2^k, where k is the number of distinct prime factors of N. Each distribution corresponds to a unique pair (a, b) such that $a \times b = N$ and $\gcd(a, b) = 1$.The number of such pairs is $2^{\omega(N)}$, where $\omega(N)$ is the number of distinct prime factors of N.Since 20! is not a perfect square, for every pair (a, b) such that $a \times b = 20!$ and $\gcd(a, b) = 1$, we have $a \neq b$. The pairs can be grouped into (a, b) and (b, a), where $a \neq b$.The number of pairs with $a < b$ is equal to the number of pairs with $b < a$.The total number of pairs is $2^{\omega(20!)}$. The number of pairs with $a < b$ is $\frac{1}{2} \times 2^{\omega(20!)} = 2^{\omega(20!)-1}$.The number of distinct prime factors of 20! is $\omega(20!) = 8$.The number of pairs (a, b) such that $a \times b = 20!$, $\gcd(a, b) = 1$, and $a < b$ is $2^{8-1} = 2^7 = 128$.Each such pair (a, b) corresponds to a unique rational number $\frac{a}{b}$ between 0 and 1 in lowest terms, with the product of the numerator and denominator equal to 20!.Final Answer: The final answer is $\boxed{128}$"}
Solution:
{ ReasoningStep: "Let the rational number between 0 and 1 be $r = \frac{a}{b}$ in lowest terms with $\gcd(a, b) = 1$, $0 < a < b$, and $a \times b = 20!$".
Tag:"Reducing a fraction to lowest terms",
Question:"In the context of finding rational numbers $r = \frac{a}{b}$ in lowest terms with $a \cdot b = 20!$, suppose you start with the unreduced fraction $\frac{7}{20!/7}$. Reduce this fraction to lowest terms.",
Solution:{
ReasoningStep: "We have the fraction $F = \frac{7}{20!/7}$. Since 20! contains 7^2 (because $\lfloor 20/7 \rfloor = 2$), the number $20!/7$ is still divisible by 7.",
Tag:"Exponent of a prime in a factorial (Legendre's formula)",
Question:"In the context of reducing the fraction 7/(20!/7), use Legendre's formula to find the exponent of the prime 7 in the prime factorization of 20!.",
Solution:" 1. Legendre's formula states that the exponent of a prime p in n! is given by the sum of $\lfloor n/p^k \rfloor fork = 1, 2, ..p^k > n.$
2. Here, n=20 and p=7:
- For k=1: $\lfloor 20/7 \rfloor = 2$
- For k=2: $\lfloor 20/49 \rfloor = 0$
- Higher powers of 7 exceed 20, so their terms are zero.
3. Summing these gives $2 + 0 = 2$."
},

{

ReasoningStep:"Compute gcd(7, 20!/7). Because 20! contains 7^2, dividing by 7 leaves one factor of 7, so gcd(7, 20!/7) = 7.",

Tag:"GCD calculation using prime factorization",

Question:"In reducing the fraction 7 one over (20! divided by 7) to lowest terms, compute gcd(7, 20! / 7) by using prime factorization.",

Solution:"1. We need the exponent of the prime 7 in 20!.

2. Use the formula for exponent of a prime p in n!: $v_7(20!) = \lfloor 20/7 \rfloor + \lfloor 20/49 \rfloor = 2 + 0 = 2$.

3. Thus 20! has a factor 7^2.

4. When we divide 20! by 7, the exponent of 7 in (20!/7) is 2 1 = 1.

5. So (20!/7) contains exactly one factor of 7, and the number 7 also contains one factor of 7.

6. Therefore gcd(7, 20!/7) = 7^1 = 7."

}

{

ReasoningStep:"Divide numerator and denominator by 7: Numerator: $7 \div 7 = 1$. Denominator: $(20!/7) \div 7 = 20!/(7^2) = 20!/49$.",

Tag:"Cancelling common factors in a fraction",

Question:"In the context of the original problem, you encounter the fraction $\frac{7}{20!/7}$. By cancelling the common factor 7 in the numerator and denominator, what is the resulting simplified denominator?",

Solution:"1. Start with the fraction $\frac{7}{20!/7}$. 2. Observe that both the numerator and denominator contain the factor 7. 3. Divide numerator and denominator by 7: - New numerator: $7 \div 7 = 1$. - New denominator: $(20!/7) \div 7 = 20!/(7^2) = 20!/49$. 4. Thus, after cancellation, the fraction becomes $\frac{1}{20!/49}$. 5. The question asks for the simplified denominator, which is $20!/49$."

},

{

ReasoningStep:"Hence the fraction in lowest terms is $\frac{1}{20!/49}$.",

Tag:"Reducing fractions to lowest terms",

Question:"Reduce the fraction 7 divided by (20! divided by 7) to lowest terms.",

Solution:"1. Start with the unreduced fraction: 7 / (20! / 7).

2. Rewrite as a single fraction: 7 × 7 / 20! = 49 / 20!.

3. Observe that 20! contains the factor $7^2 = 49, sogcd(49, 20!) = 49$.

4. Divide numerator and denominator by 49: • Numerator: 49 ÷ 49 = 1 • Denominator: 20! ÷ 495. Hence the fraction in lowest terms is 1 / (20! / 49).

6. Compute 20! = 2 432 902 008 176 640 000, then divide by 49 to get 49 651 061 391 360 000.

7. Final reduced form is 1 / 49651061391360000.",],

{

ReasoningStep:"Set $N = 20!$. Its prime factorization has $k = 8$ distinct primes (2, 3, 5, 7, 11, 13, 17, 19).",

Tag:"Prime Factorization",

Question:"How many distinct prime numbers appear in the prime factorization of 20!?",

Solution:

{ ReasoningStep:"By definition, 20! = 1×2×3×...×20, so every prime p 20 divides 20!.",

Tag:"Definition of factorial",

Question:"In the definition of factorial, 20! is written as the product 1×2×3×...×20. How many integers are multiplied together in this product?",

Solution:"By definition, n! = 1×2×3×...×n, which multiplies all integers from 1 up to n. For n = 20, the integers are 1, 2, 3, ..., 20. Counting these gives 20 numbers being multiplied.",

{ ReasoningStep:"The primes less than or equal to 20 are 2, 3, 5, 7, 11, 13, 17, and 19, and there are no others in that range.",

Tag:"Primality Testing",

Question:"Among the integers from 1 through 20 inclusive, how many are prime numbers?",

Solution:"1. By definition, a prime number is an integer greater than 1 that has no positive divisors other than 1 and itself.

2. We test each integer from 1 to 20: - 1 is not prime by definition. - 2 is divisible only by 1 and 2 → prime. - 3 is divisible only by 1 and 3 → prime. - 4 is divisible by 1, 2, 4 → not prime. - 5 is divisible only by 1 and 5 → prime. - 6 is divisible by 1, 2, 3, 6 → not prime. - 7 is divisible only by 1 and 7 → prime. - 8 is divisible by 1, 2, 4, 8 → not prime. - 9 is divisible by 1, 3, 9 → not prime. - 10 is divisible by 1, 2, 5, 10 → not prime. - 11 is divisible only by 1 and 11 → prime. - 12 is divisible by 1, 2, 3, 4, 6, 12 → not prime. - 13 is divisible only by 1 and 13 → prime. - 14 is divisible by 1, 2, 7, 14 → not prime. - 15 is divisible by 1, 3, 5, 15 → not prime. - 16 is divisible by 1, 2, 4, 8, 16 → not prime. - 17 is divisible only by 1 and 17 → prime. - 18 is divisible by 1, 2, 3, 6, 9, 18 → not prime. - 19 is divisible only by 1 and 19 → prime. - 20 is divisible by 1, 2, 4, 5, 10, 20 → not prime.

3. The primes in this range are 2, 3, 5, 7, 11, 13, 17, and 19.

4. Counting them gives a total of 8 primes.",

{ ReasoningStep:"Therefore, the prime factorization of 20! includes exactly these 8 distinct primes.",

Tag:"Prime Factorization",

Question:"When prime factorizing 20!, we include every prime number that is less than or equal to 20. How many distinct prime numbers appear in the prime factorization of 20! ?",

Solution:"1. A prime divides 20! exactly if it is 20, since 20! is the product of all integers from 1 to 20.

2. The primes 20 are 2, 3, 5, 7, 11, 13, 17, and 19.

3. Counting these gives 8 distinct primes in the factorization of 20!." },

{ ReasoningStep:"Since $\gcd(a, b) = 1$, each prime power $p_i^{e_i}$ of N must go entirely to a or b, giving 2^k pairs (a, b).",

Tag:"Fundamental Theorem of Arithmetic",

Question:"Let N be 20! and write N in its prime factorization $\mathrm{N} = \mathrm{p}_1^{e_1} \cdot p_2^{e_2} \cdot \ldots \cdot p_k^{e_k}$.

Using the Fundamental Theorem of Arithmetic, in how many ways can you choose a pair of positive integers (a, b) such that ab = N and gcd(a, b) = 1 by assigning each prime power $p_i^{e_i}$ entirely to either a or b?",

Solution:{ReasoningStep:"By the Fundamental Theorem of Arithmetic, 20! factors into prime powers corresponding to each prime p 20.",

Tag:"Prime Factorization",

Question:"In the prime factorization of 20!, what is the exponent of the prime 2?",

Solution:"1. By Legendre's formula, the exponent of a prime p in n! is $\lfloor n/p \rfloor + \lfloor n/p^2 \rfloor + \lfloor n/p^3 \rfloor + \ldots$ until the terms are zero.

2. For p = 2 and n = 20: $\lfloor 20/2 \rfloor = 10 \lfloor 20/4 \rfloor = 5 \lfloor 20/8 \rfloor = 2 \lfloor 20/16 \rfloor = 1 \lfloor 20/32 \rfloor = 0$ (and all higher powers give zero)

3. Sum these: 10 + 5 + 2 + 1 = 18.",

},

{ReasoningStep:"The primes 20 are 2, 3, 5, 7, 11, 13, 17, 19, so there are k = 8 distinct prime powers in the factorization of 20!.",

Tag:"Listing primes up to a given limit",

Question:"In the prime factorization of 20!, how many distinct prime bases (i.e., prime numbers 20) appear?",

Solution:"

1. To find the distinct prime factors of 20!, we list all prime numbers that are less than or equal to 20.

2. A prime number is a number greater than 1 with no positive divisors other than 1 and itself.

3. Checking the integers from 2 up to 20, the primes are: 2, 3, 5, 7, 11, 13, 17, 194. Count these primes: there are 8 of them.",

},

{ReasoningStep:"To form a pair (a, b) with ab = N and gcd(a, b) = 1, each prime power $p_i^{e_i}$ must go entirely to a or entirely to b.",

Tag:"Fundamental Theorem of Arithmetic",

Question:"Let N = 20! and write its prime factorization as $N = p_1^{e_1} \cdot p_2^{e_2} \cdot \ldots \cdot p_k^{e_k}$. By the Fundamental Theorem of Arithmetic, to form pairs of positive integers (a, b) with ab = N and gcd(a, b) = 1, each prime power $\mathrm{p}_i^{e_i} must go entirely to a or entirely to b. For a single prime power \mathrm{p}_i^{e_i}$, in how many ways can you assign it to either a or b so that gcd(a, b) = 1?",

Solution:" 1. The Fundamental Theorem of Arithmetic ensures N's prime factorization is unique.

2. To keep gcd(a, b) = 1, you cannot split $p_i^{e_i}$ between a and b.

3. Therefore, for that prime power, you have exactly two choices: - Assign $p_i^{e_i}$ to a (so exponent in a is $\mathrm{e}_i, in b is 0), - or assign \mathrm{p}_i^{e_i}$ to b (so exponent in a is 0, in b is e_i).

4. No other distributions are allowed without violating gcd(a, b) = 1.",

},

{ReasoningStep:"Each of the k = 8 prime powers has 2 choices, so by the multiplication principle there are $2^k = 2^8$ ways.",

Tag:"Fundamental Counting Principle",

Question:"Suppose the prime factorization of 20! contains 8 distinct prime powers. If you assign each prime power entirely to either integer a or integer b, in how many ways can you make these assignments by the Fundamental Counting Principle?",

Solution:"1. There are 8 prime powers to assign.

2. Each prime power has 2 independent choices: assign it to a or to b.

3. By the Fundamental Counting Principle, multiply the number of choices for each prime power: Total ways = 2•2•…•2(8 times) = 2^8 = 256.",

}],

},

{ReasoningStep:"As 20! is not a perfect square, $a \neq b$, so half the pairs have $a < b$, yielding $2^{k-1} = 2^7 = 128$.",

Tag:"Divisor Pairing Principle",

Question:"Using the Divisor Pairing Principle, in how many ways can we write 20! as a product ab of two positive integers with gcd(a,b)=1 and a<b?",

Solution:

{ReasoningStep:"The prime factorization of 20! involves exactly k = 8 distinct primes (2, 3, 5, 7, 11, 13, 17, 19).",

Tag:"Prime factorization",

Question:"In the prime factorization of 20!, how many distinct prime factors does it contain?",

Solution:"1. By definition, 20! = 1·2 · 3 · … · 20.

2. Every prime p 20 divides one of the factors in the product.

3. The prime numbers less than or equal to 20 are 2, 3, 5, 7, 11, 13, 17, and 19.

4. There are 8 such primes.",

},

{ReasoningStep:"To have ab = 20! and gcd(a, b) = 1, each prime's entire power in 20! must go either to a or to b.",

Tag:"Unique Prime Factorization",

Question:"In the prime factorization of 20!, what is the exponent of the prime 3?",

Solution:"1. By unique prime factorization, the exponent of a prime p in n! is given by summing $\lfloor n/p^k \rfloor fork \geq 1 until p^k > n.$

2. For p=3 and n=20: – $\lfloor 20/3 \rfloor = 6\lambda\lfloor 20/9 \rfloor = 2\lambda\lfloor 20/27 \rfloor = 0(stophere)$

3. Sum of these is 6 + 2 = 8.",

},

{ReasoningStep:"Therefore there are $2^k = 2^8 = 256$ unordered assignments of prime-powers to (a, b).",

Tag:"Fundamental Counting Principle",

Question:"The prime factorization of 20! involves 8 distinct prime-power factors. Suppose each entire prime-power factor must be assigned either to integer a or to integer b. Using the Fundamental Counting Principle, in how many ways can these 8 prime powers be distributed between a and b?",

Solution:"1. There are 8 distinct prime-power factors in 20! (for primes 2, 3, 5, 7, 11, 13, 17, 19).

2. For each prime-power factor, we have exactly 2 choices: assign it to a or assign it to b.

3. By the Fundamental Counting Principle, the total number of ways to make all choices is 2 × 2 × … × 2 (8 factors) = 2^8.

4. Compute $2^8 = 256$.",

},

{ReasoningStep:"Since 20! is not a perfect square, no assignment yields a = b, so exactly half of these yield a < b, giving 256/2 = 128.",

Tag:"Symmetry argument in combinatorial counting",

Question:"Suppose there are 256 ordered pairs of positive integers (a,b) such that ab = 20! and gcd(a,b) = 1. Using a symmetry argument, how many of these pairs satisfy a < b?",

Solution:"1. We are given that there are 256 ordered coprime factor pairs (a,b) with ab = 20!.

2. For each ordered pair (a,b), there is a corresponding "swapped" pair (b,a).

3. Because 20! is not a perfect square, no pair has a = b; every pair is distinct from its swap.

4. Thus the 256 ordered pairs split evenly into two groups: those with a < b and those with a > b.

5. By symmetry, the number with a < b is half of 256, namely 256/2 = 128.",}, }